DISCRETE MATHEMATICS I

B.MATH 2ND YEAR

MID TERM EXAM

Instructions

- Part A contains 9 questions. Each question carries 5 marks each. Answer any 6.
- Part B contains 3 questions. Each question carries 10 marks each. Answer any 2.
- Time limit for the exam is 3 hours.
- You are allowed to name/quote and use any theorem, proposition, lemma or corollary proved in class.
- You are not allowed to quote and use problems discussed in class, assignments and quizzes without proof.

Notations

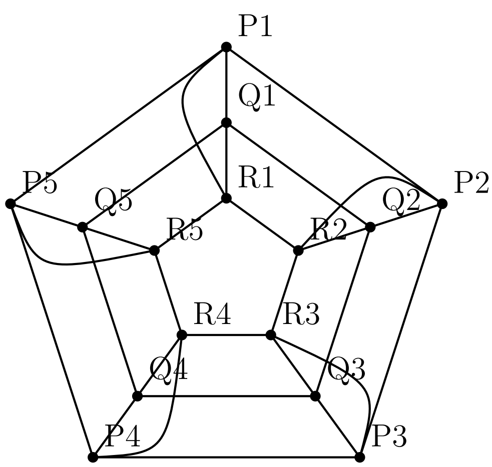
- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $[n] = \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$.
- Set of integers is $\mathbb{Z} = \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$
- A positive integer is an element of the set $\mathbb{N} \setminus \{0\}$.
- Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then $\displaystyle\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$
- The sequence of Fibonacci numbers $\{F_n\}_{n\in\mathbb{N}}$ is defined by the recurrence relation
$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \;\; \forall \; n \in \mathbb{N}.$$
The ordinary generating function of $\{F_n\}_{n\in\mathbb{N}}$ is $\dfrac{x}{1 - x - x^2}$.
- A composition of n is a sequence $(a_1, a_2, \ldots, a_k)$ of positive integers such that $k \geq 0$ and $\displaystyle\sum_{i=1}^{k} a_i = n$. The numbers a_i are called parts of the composition. *Note that, if $k = 0$, then the sequence is an empty sequence, and in that case, $\displaystyle\sum_{i=1}^{k} a_i = 0$.*

PART A (5 MARKS PER QUESTION, ANSWER ANY 6)

1. How many ways can one place 8 rooks on black squares of a 8×8 chessboard such that no two can attack each other?
Recall that a rook can move an arbitrary number of cells along the row or column it is on (either row or column in a single move).

2. How many ordered triples (A, B, C) of subsets of an n-element set S satisfy $A \cup B \cup C = S$?

3. Consider the following simple undirected graph G :



(a) Does G contain a closed Eulerian trail?

(b) Prove that G contains a Hamiltonian cycle.

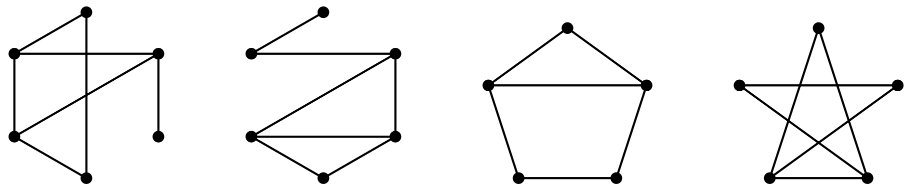
4. 61 students are taking the Discrete Mathematics midterm examination. Every student is copying from exactly 3 students each. Is it possible for every student to only be copying from students who are copying from them?

5. Prove the following identity for every $n \geq 1$:

$$\sum_{k \geq 1} \binom{k}{n-k} = F_{n+1}$$

Where F_k is the k-th Fibonacci number.

6. Consider the following four simple undirected graphs :

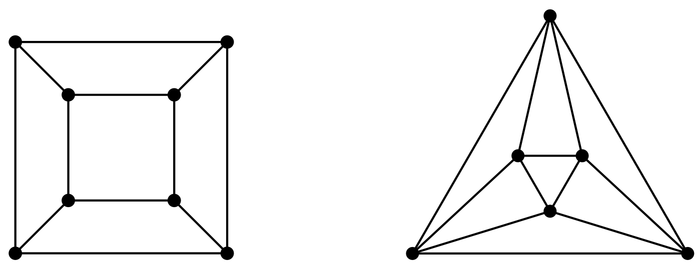


. G_1 G_2 G_3 G_4

(a) Is G_1 isomorphic to G_2? Justify your answer.

(b) Is G_3 isomorphic to G_4? Justify your answer.

7. Are the following two graphs bipartite? If not, find their chromatic numbers.



8. Let $G = (V, E)$ be a connected finite undirected graph. Let $e \in E$ be an edge. Consider the graph G_1 obtained by deleting the edge e from G, i.e., $G_1 = (V, E \setminus \{e\})$. What is the maximum possible number of connected components of G_1?

9. Eight students took a test with 39 one-mark questions (with no negative or partial marking). Each of them got a positive score. Prove that one can always find two disjoint nonempty groups of students whose scores add up to the same number.

Part B (10 marks per question, answer any 2)

10. (a) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$a_0 = 2, \qquad a_{n+1} = 2a_n - 3 \ \forall\ n \in \mathbb{N}.$$

Describe the ordinary generating function of the sequence $\{a_n\}_{n\in\mathbb{N}}$.

(b) Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence defined by the recurrence relation

$$c_0 = 3, \quad c_1 = 7, \quad c_n = 5c_{n-1} - 6c_{n-2} \ \forall\ n \geq 2.$$

Find the value of c_n for all $n \in \mathbb{N}$.

11. (a) Let n be a positive integer. Prove that, given a square piece of paper, one can cut it into $8n + 1$ smaller squares (Not necessarily of the same size).

(b) Let $m \geq 15$ be a positive integer. Prove that, given a square piece of paper, one can cut it into m smaller squares (Not necessarily of the same size).

12. (a) Let $n \in \mathbb{N}$. Prove that the number of compositions of n with every part 1 or 2 is F_{n+1}.

(b) Let $n \in \mathbb{N} \setminus \{0\}$. Prove that the number of compositions of n with every part odd is F_n.

Here F_k is defined to be the k-th Fibonacci number.